Journal of Sound and Vibration 329 (2010) 2792–2808

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Minimum stiffness location of point support for control of fundamental natural frequency of rectangular plate by Rayleigh–Ritz method

D. Wang *, Z.C. Yang, Z.G. Yu

Department of Aeronautical Structural Engineering, Northwestern Polytechnical University, P.O. Box 118#, Xi'an, Shaanxi 710072, PR China

ARTICLE INFO

Article history:
Received 15 May 2009
Received in revised form
19 January 2010
Accepted 28 January 2010
Handling Editor: M.P. Cartmell
Available online 23 February 2010

ABSTRACT

The Rayleigh–Ritz method is employed to determine the minimum stiffness location of the elastic point support for raising the fundamental natural frequency of a rectangular plate to the second frequency of the unsupported plate, which usually is the upper limit of the first frequency for a single support. Based on the optimal design of the support position, the minimum stiffness can be obtained numerically by solving a characteristic eigenvalue sub-problem. In the Rayleigh–Ritz procedure the boundary characteristic orthogonal polynomials are used for the admissible functions. Several typical examples of plate structures with the additional point supports are analyzed in detail, and the results prove that the proposed method is very effective in the solution to the optimal design of the supports. It will be shown that elastic supports can be designed like rigid ones for the purpose of increasing the fundamental natural frequency of a rectangular plate.

1. Introduction

Plates with point (or simple) supports are commonly found in civil, aeronautical and marine engineering. Usually, supports are used to hold the structure properly and, at the same time, alter the structural characteristic properties. Thus far, a great number of publications are available in the literature on the analysis of plates with point supports at any locations. In general, an exact solution to the free vibration of a thin rectangular plate with general (elastic or rigid) point supports is not available. Therefore, there have been many numerical approaches, such as the Rayleigh–Ritz method [1–7] (or sometimes called the Ritz method [8]), the superposition method [9,10], the flexibility function approach [11], the finite strip method [12,13], the finite element method (FEM) [14,15], etc., applied in order to determine the natural frequencies and vibration modes of the plate. Within the existing approaches, the Rayleigh–Ritz method in conjunction with the Lagrange multiplier technique has been broadly adopted to constitute the characteristic frequency equation of the structure. In the procedure, a series of admissible displacement functions, e.g. the ordinary or orthogonal polynomials [1–5], static beam functions [6], trigonometric functions [7] and so on is typically chosen to represent the transverse deflection of the plate. By means of the minimization of the total energy functional, involving the potential and kinetic energies, of the system, the natural frequencies and mode shapes of the structure can then be obtained numerically.

It has long been known that a support, except for preventing a structure from motion or excessive deflection, can also be utilized to control the structural vibration behaviors. For example, it can be used to raise a natural frequency or to modify

* Corresponding author. Tel.: +86 29 88493386; fax: +86 29 88460589.
E-mail address: wangdng66@yahoo.com.cn (D. Wang).

0022-460X/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2010.01.034

the vibration pattern corresponding to a given order of a natural frequency. Narita [1] demonstrated that the dynamic performances of a cantilever plate can be significantly changed with use of a point constraint, i.e. a point support with an infinite translational stiffness against the deflection of the plate. Moreover, it was also observed that a natural frequency of the plate can often be increased up to the maximum by varying the position of a rigid point support. Nevertheless, as no support is absolutely rigid to forbid completely the lateral motion at the support point, the zero deflection constraint cannot be achieved physically. Consequently, the resulting solutions on the basis of the point constraint are limited in practical applications. Actually, from the existing computational results [12] it was shown clearly that a rigid point support was not necessary to increase any frequency of the plate up to its maximum by varying the support location.

In recent years, considerable attention has been paid to the optimal design of the position and stiffness of the interior point supports for control of the fundamental natural frequency of a rectangular plate [14–17]. As is well known, a flexible point support may increase the structural lowest natural frequency to between the first and the second ones of the original, unsupported system (with no additional support). Furthermore, Won and Park [14], Friswell and Wang [16] have demonstrated that when using a single flexible support to increase the fundamental natural frequency to its upper limit of the original second natural frequency, there exists a certain critical (minimum) value of the support stiffness provided that it is positioned appropriately. Above this minimum stiffness, the support cannot increase the lowest natural frequency any further due to the very mode switching. In other words, an elastic support with a finite stiffness can achieve the similar result to a rigid support when it is located optimally. This minimum stiffness is virtually of great importance in a practical support design. Moreover, Wang et al. [15] showed that when the support stiffness is greater than the critical value, the optimal location is no longer determined uniquely, but is in a suitable region. Thus far, the problem of the optimal location and minimum stiffness of a point support is studied mostly with the FEM [14–16]. The primary objective of this study is to develop an alternative numerical formulation based on the Rayleigh–Ritz procedure to gain both the optimal location and the minimum support stiffness for control of the fundamental natural frequency of a rectangular plate. The finite element process, in essence, can be understood to be the Rayleigh–Ritz procedure. The technical difference is only the basis displacement approximation (or shape functions) assumed within the suitable region in each of the processes and then the number of the degrees of freedom. As a result, the present method is more efficient.

A rectangular plate having one or two edges conventionally restrained (simply supported or clamped) along with one or several additional flexible supports is a typical model for dynamic analysis of the structural behaviours. When designing a support, such as a column of a slab in civil engineering or for a printed circuit board in electrical engineering, it is of particular interest to know exactly the optimal position as well as the minimum stiffness. Support position optimization has now been an important problem in the related field. In this work, a general approach is presented for the solution of the optimal position and the minimum stiffness of one or more elastic supports in order to increase the fundamental natural frequency of a rectangular plate. The Kirchhoff plate theory is adopted for the structural dynamic analysis. The boundary characteristic orthogonal polynomials, which have gained some popularity owing to its ease of generation and manipulation [2–5], are exploited herein in the Rayleigh–Ritz approach to formulate the admissible displacements. In the solution process, the requirement conditions of the optimal support position are dealt as the optimization objective, and the minimum stiffness of the support is determined by numerically solving a characteristic eigenvalue sub-problem. Several typical examples of different boundary conditions and aspect ratios of the plate are employed to illustrate the solutions for the optimal support position and minimum stiffness and to make a comparison with the results achieved previously by the FEM [15,16] for the purpose of verification of the developed method. Meanwhile, some representative vibration mode shapes of the plate supported by the additional simple support with the minimum stiffness gained are plotted to verify the results. It will be seen that the present method is very effective for the optimal layout design of the flexible support in the frequency control problem.

2. Generation of boundary characteristic orthogonal polynomials

Assume that a thin rectangular plate lying in the x–y plane, as shown in Fig. 1, undergoes free flexural vibration. In the Rayleigh–Ritz approach the transverse deflection $W(x, y)$ may be expressed as

$$W = \sum_{m,n=1}^{M,N} A_{mn}\phi_m(x)\psi_n(y) \tag{1}$$

in which $\phi_m(x)$ and $\psi_n(y)$ denote the assumed admissible functions in the x- and y-directions, respectively, which satisfies at least the geometrical boundary conditions of the plate. M and N are the numbers of the trial functions adopted in each of the directions. Thus, it remains to determine the unknown coefficients A_{mn} in the deflection expression to satisfy the vibration equation and the natural boundary conditions. Apparently, the accuracy and convergence of the solution depend highly upon the choice of the admissible displacement functions. In this context, the boundary characteristic orthogonal polynomials suggested by Bhat [2] are utilized. In the displacement series in Eq. (1), the starting function is designed to satisfy both the geometrical and natural boundary conditions of the corresponding beam, whereas each of the subsequent functions, generated in a recurrence formula, meets only the geometrical boundary conditions. Consequently, the natural boundary conditions of the plate will be satisfied by summation of all the terms.

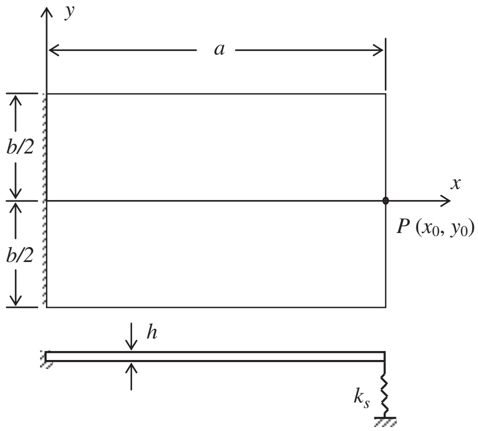

Fig. 1. A uniform rectangular plate of one edge conventionally restrained is additionally supported by an elastic point support.

Table 1
The starting function coefficients, integration interval and generating function for a beam with various combinations of the boundary edges clamped (C), simply supported (S) and free (F).

Boundary conditions	Integration intervals	$f(x)$	Coefficients				
			R_1	R_2	R_3	R_4	R_5
F–F	−0.5–0.5	$x+0.5$	1	0	0	0	0
S–F	0–1	x	0	1	0	0	0
C–F	0–1	x	0	0	6	−4	1

As a preliminary step, this section briefly describes the generation of boundary characteristic orthogonal polynomials. Given a polynomial $\phi_1(x)$, a set of the orthogonal polynomials in the interval $a \leqslant x \leqslant b$ can be generated recursively by using a Gram–Schmidt process [2–4],

$$\phi_k(x) = [f(x) - B_k]\phi_{k-1}(x) - C_k\phi_{k-2}(x) \tag{2a}$$

$$B_k = \int_a^b f(x)w(x)\phi_{k-1}^2(x)\,\mathrm{d}x \Big/ \int_a^b w(x)\phi_{k-1}^2(x)\,\mathrm{d}x \tag{2b}$$

$$C_k = \int_a^b f(x)w(x)\phi_{k-1}(x)\phi_{k-2}\,\mathrm{d}x \Big/ \int_a^b w(x)\phi_{k-2}^2(x)\,\mathrm{d}x \tag{2c}$$

where $w(x)$ is a weighting function chosen as unity for an isotropic plate of uniform thickness, and $f(x)$ is a generating function chosen to ensure the satisfaction of the geometrical boundary conditions for the higher order functions. In this work, it is taken just as one power of x for the lower orders of the polynomials generated afterwards. $\phi_0(x)$ is defined as zero. The polynomial $\phi_k(x)$ satisfies the orthogonality condition

$$\int_a^b w(x)\phi_k(x)\phi_l(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } k \neq l \\ B_{kl} & \text{if } k = l \end{cases} \tag{3}$$

where the value of B_{kl} depends upon the normalization of $\phi_k(x)$ used in the procedure. In the authors' experience, B_{kl} being always equal to unity may result in much numerical instability in the solution process. The starting function $\phi_1(x)$ is constructed so as to satisfy both the geometrical and natural boundary conditions of the equivalent beam problems [2]. For a beam of length l (=$b-a$), the starting function may thus, in a generalized form, be written as [4]

$$\phi_1(x) = \sum_{i=1}^{5} R_i (x/l)^{i-1} \tag{4}$$

in which the coefficients R_i are selected so as to meet the equivalent beam's boundary conditions. Table 1 offers the coefficients R_i and the integration intervals for combinations of different boundary edges of the plate to be studied ahead. Likewise, the shape function in the y-direction $\psi_n(y)$ can be generated following the above procedure.

3. Requirement of an optimal support position

It has been demonstrated that when located optimally, a support can raise a natural frequency of a plate to a certain value with the minimum stiffness [14–16]. To determine the optimal position of an elastic support, it is a basic prerequisite to estimate the derivative of a natural frequency with respect to the location of a point support. With the quantitative sensitivity information, both the search direction and the final optimal position of a support can be determined accurately. The derivative of a natural frequency of a plate with respect to the position of a simple support, which does not offer a resistance to the rotation of the plate, has already been developed with the discrete method [15]. For a natural frequency of the plate, it gives

$$\left.\frac{\partial \omega_i^2}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} = -2kW_i(x_0,y_0)\theta_{yi}(x_0,y_0) \tag{5a}$$

$$\left.\frac{\partial \omega_i^2}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} = 2kW_i(x_0,y_0)\theta_{xi}(x_0,y_0) \tag{5b}$$

where the subscript i indicates the order of a natural frequency of the plate. $W_i(x_0, y_0)$ is the transverse deflection of the related vibration mode mass-normalized, and both $\theta_{xi}(x_0, y_0)$ and $\theta_{yi}(x_0, y_0)$ are the rotations (or slopes) about the x- and y-axis, respectively, at the support point. Apparently, the satisfaction of the optimal position of an additional support requires the derivative of a natural frequency vanishing, which can be accomplished by either $W_i(x_0, y_0)=0$ or $\theta_{xi}=\theta_{yi}=0$. Note that the zero deflection is usually of little interest in the current problem since this requires the position of a flexible support just at a node of a mode, which, in fact, does not raise the related frequency at all. In other words, once the slopes of the vibration mode equal zeroes at the support point, that is,

$$\theta_{xi}(x_0,y_0) = \left.\frac{\partial W_i}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} = \sum_{m,n=1}^{M,N} A_{mn}\phi_m(x_0)\psi_n'(y_0) = 0 \tag{6a}$$

$$\theta_{yi}(x_0,y_0) = -\left.\frac{\partial W_i}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} = -\sum_{m,n=1}^{M,N} A_{mn}\phi_m'(x_0)\psi_n(y_0) = 0 \tag{6b}$$

an elastic support may raise the natural frequency of the plate to a certain value with the minimum stiffness required. For example, when raising the lowest natural frequency ($i=1$) to a maximum, the optimal position of the support must be where the first mode shape has zero values of the slopes along the two coordinate axes. In this application, the zero-slope condition for an optimal design of a support position is not imposed by each term of the polynomials in Eq. (1), but is fulfilled by the summation of the series.

On the other hand, the zero-slope conditions for the optimum design of a point support imply a partially clamped point with use of an ordinary point support. This may be the particular reason why only the translational stiffness is taken account, while the rotational stiffnesses are neglected in the previous studies for support optimizations [14–16]. However, for a larger or smaller value than the minimum support stiffnesses, the slopes of the mode shape would not be zero at the support point, and the support would remain a simple support.

4. Formulation of characteristic eigenvalue equation

The minimization of the total energy functional of a vibration system in accordance with the Rayleigh–Ritz principle will generate the characteristic frequency equation of a plate with flexible supports. Herein the Lagrange multipliers are not applied to enforce the additional requirements for the optimal position of a support in Eq. (6) since this may lead to some difficulties in extracting the eigenvalue. However, these requirements will be sought as the objective. For a rectangular plate of the dimensions shown in Fig. 1 along with one transverse elastic point support the maximum potential energy U stored in the plate due to bending and twisting and in the support due to axial deformation is estimated by

$$U = \frac{D}{2}\int_0^a\int_{-b/2}^{b/2}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}k_s W^2(x_0,y_0) \tag{7}$$

and the maximum kinetic energy T is

$$T = \frac{\rho h\omega^2}{2}\int_0^a\int_{-b/2}^{b/2} W^2\,\mathrm{d}x\,\mathrm{d}y \tag{8}$$

where ω is the natural frequency of harmonic vibration in radians per second, ρ the mass density per unit volume and h the uniform thickness of the plate. $D=Eh^3/12(1-\nu^2)$ is the constant flexural rigidity of a plate, E is Young's modulus and ν is Poisson's ratio of the material, which is taken as 0.3 in the subsequent numerical examples of isotropic plates.

After substituting the plate displacement function in Eq. (1) into the above energy expressions, a set of $m\times n$ homogeneous equations of A_{mn} is then formulated by differentiating the Lagrangian energy, defined by U–T, with regard to each of the undetermined coefficient A_{mn}. The generalized eigenvalue problem is

$$[\mathbf{K}_p - \lambda^2 \mathbf{M}_p + \gamma_s \mathbf{K}_s]\{A\} = 0 \tag{9a}$$

where

$$\begin{aligned}[k]_p = &\int_0^1 \phi_m'' \ \phi_i'' \ \mathrm{d}\xi \int_{-1/2}^{1/2} \psi_n \psi_j \, \mathrm{d}\eta + \nu\alpha^2 \Big(\int_0^1 \phi_m \phi_i'' \ \mathrm{d}\xi \int_{-1/2}^{1/2} \psi_n'' \ \psi_j \, \mathrm{d}\eta + \int_0^1 \phi_m'' \ \phi_i \, \mathrm{d}\xi \int_{-1/2}^{1/2} \psi_n \psi_j'' \ \mathrm{d}\eta\Big) \\ &+ \alpha^4 \int_0^1 \phi_m \phi_i \ \mathrm{d}\xi \int_{-1/2}^{1/2} \psi_n'' \ \psi_j'' \ \mathrm{d}\eta + 2(1-\nu)\alpha^2 \int_0^1 \phi_m' \ \phi_i' \ \mathrm{d}\xi \int_{-1/2}^{1/2} \psi_n' \ \psi_j' \ \mathrm{d}\eta \end{aligned} \tag{9b}$$

$$[m]_p = \int_0^1 \phi_m \phi_i \, \mathrm{d}\xi \int_{-1/2}^{1/2} \psi_n \psi_j \, \mathrm{d}\eta \tag{9c}$$

$$[k]_s = \alpha \phi_m(\xi_0) \psi_n(\eta_0) \phi_i(\xi_0) \psi_j(\eta_0) \tag{9d}$$

in which, for generality and convenience, introduced are the nondimensional coordinates with respect to the plate planar dimensions $\xi = x/a$, $\eta = y/b$, aspect ratio $\alpha = a/b$, natural frequency parameter $\lambda = \omega a^2 \sqrt{\rho h / D}$ and normalized support stiffness $\gamma_s = k_s a^2 / D$. The prime denotes differentiation with respect to ξ or η for each of the admissible functions, respectively. All of that introduction will give some computational advantages for the standard solution of the eigenvalue problem for γ_s. Note that the rank of the equivalent stiffness matrix, $\mathbf{K}_s$ is equal to one for one flexible support, but will be the number for multiple supports supposed that all supports have the same stiffness [16]. From Eq. (9d) it is clear that $\mathbf{K}_s$ relies closely upon the support location. Let the support position $P(\xi_0, \eta_0)$ and the frequency parameter λ_d desired be given a priori, it is therefore possible to calculate all elements of the matrices in Eq. (9), and then a characteristic eigenvalue problem is formulated as

$$[\mathbf{K}_p - \lambda_d^2 \mathbf{M}_p]\{A\} = \gamma_s(-\mathbf{K}_s)\{A\} \tag{10}$$

The minimum positive eigenvalue for γ_s in Eq. (10) is virtually the minimum support stiffness required to raise the plate natural frequency to the frequency specified. Moreover, the associated eigenvector $\{A\}$ is the solution of the coefficients A_{mn}, with which the corresponding mode shape of the plate supported can be obtained upon substitution of the solution into Eq. (1). Furthermore, the slopes at the support can be computed by using Eq. (6). Note that these values, in general, can only be attained numerically.

Friswell and Wang [16] solved the minimum support stiffness by transformation of the eigenvalue problem of Eq. (9a) into an equivalent sub-problem of much smaller dimension in Eq. (10). This procedure can also be utilized in the present work for estimation of the minimum support stiffness γ_s. Suppose that the dynamic stiffness matrix of the plate $\mathbf{D}_p = \mathbf{K}_p - \lambda_d^2 \mathbf{M}_p$ is non-singular and $\mathbf{K}_s = \mathbf{P}_r \mathbf{P}_r^T$. Then, the characteristic determinant equation from Eq. (9a) becomes

$$\det[\mathbf{I}_r + \gamma_s \mathbf{P}^T \mathbf{D}_p^{-1} \mathbf{P}] = \mathbf{0} \tag{11}$$

where r is the rank of the support stiffness matrix. $\mathbf{I}_r$ is the $r \times r$ identity matrix. In the case of a single support, r=1, and

$$\mathbf{P} = \left[\begin{matrix} \phi_1(\xi_0)\psi_1(\eta_0) & \phi_1(\xi_0)\psi_2(\eta_0) & \cdots & \phi_m(\xi_0)\psi_{n-1}(\eta_0) & \phi_m(\xi_0)\psi_n(\eta_0) \end{matrix}\right]^T \tag{12}$$

Thus, the support stiffness required is immediately obtained from the resulting characteristic determinant

$$\gamma_s = -\frac{1}{\mathbf{P}^T \mathbf{D}_p^{-1} \mathbf{P}} \tag{13}$$

To identify the optimal support location a numerical procedure may be used based on the design derivatives in Eq. (5). Wang et al. [15] presented a heuristic optimization procedure to find out the optimal support location by achieving null frequency derivatives at the support point of the fundamental mode shape. Alternatively, the optimal location is determined by minimizing the support stiffness in combination with slopes as a single objective function

$$J(x_0, y_0) = (\theta_{xi}(x_0, y_0))^2 + (\theta_{yi}(x_0, y_0))^2 \tag{14}$$

It is worth mentioning that this objective function is a weak objective. It is not enforced to be zero in the design optimization process since the optimal solution is often at the bimodal state [14]. Sometimes, the optimal locations of the supports are just at the nodal lines of the correlated mode shape with mode switching. In this situation, the derivative of the fundamental natural frequency is still zero although the slopes may be nonvanishing.

5. Numerical results

To illustrate the capability of the presented procedure for obtaining the minimum stiffness and optimal position of point supports, several examples of the rectangular plate of different boundary conditions are analyzed in order to increase their respective lowest natural frequency, and the results are compared with those obtained before by FEM [15,16]. As may be expected, an excellent agreement of the results can be obtained between these two approaches.

Herein, both M and N are equal to 15 for the admissible polynomials along the two axes, respectively. More terms in the deflection series may, sometimes, make the eigenvalue problem ill-conditioned, and induce formidable computational problems in the solution process.

5.1. One edge of the rectangular plate restrained

5.1.1. One elastic support on the free edge opposite to the restrained edge

A rectangular plate with only one boundary edge clamped or simply supported and others are free is one of the most often used models in engineering analysis. Assume the y-axis along the restrained edge and the x-axis coincident with the centerline of the rectangular plate, as indicated in Fig. 1. An elastic point support lying along the free edge opposite to the restrained edge is used to raise the fundamental natural frequency of the plate. Obviously, all the vibration modes must be either symmetric or antisymmetric with respect to the x-axis. Due to this characteristic, the zero-slope requirement of the lowest vibration mode (corresponding to the first bending mode) at the support position in the y-direction can be satisfied readily with the support being located at the mid-point of the free edge. Because the support is right on the nodal line of the second vibration mode (corresponding to the first torsional mode) of the unsupported structure, it is then recognized that the first natural frequency can only be raised to its upper limit of the original second natural frequency even with a rigid support [1]. In fact, an elastic support with a certain value of the minimum stiffness can also raise the lowest natural frequency of the plate to this upper limit. Afterwards, increase in the stiffness of the elastic support above this critical stiffness cannot increase the fundamental natural frequency any further due to mode switching.

The computational results of the first three natural frequency parameters for the unsupported plate are given in Table 2 with different aspect ratios of 1 (a square plate) and 1.5, respectively, by using the Rayleigh–Ritz method. Simultaneously, results by FEM [16] are also listed for comparison. Evidently, the two numerical procedures offer almost identical frequency values. Table 3 provides the minimum support stiffness calculated for the rectangular plate and the corresponding first three natural frequency parameters of the plate with the optimal point support design. As is expected, the minimum support stiffness depends highly upon the aspect ratio α and the restrained condition of the plate boundary edge. With increase in the aspect ratio, the minimum support stiffness increases remarkably for the cantilever plate. In addition, it is found that the minimum stiffness for a square plate with an edge simply supported is larger than that with an edge clamped. This is because the former frequency increment is larger than the latter. Fig. 2 shows the two fundamental mode shapes of the square plate restrained differently with a minimum stiffness support. By using an elastic support, the fundamental natural frequency arrives at its upper limit of the original second natural frequency, which is usually of an antisymmetric mode shape about the centerline. Note that the fundamental frequency is now doubly repeated, and in Fig. 2 only shown are the symmetric mode shapes. Besides, the results computed by the FEM [16] are also listed in Table 3, and very good agreement between these two approaches can be observed in all cases.

It should be pointed out that for the plate of a simply supported edge with aspect ratio α=1.5, there is no positive eigenvalue in the eigenvalue solution process, which implies that even a rigid point support cannot raise the first bending

Table 2
The first three natural frequency parameters of an unsupported rectangular plate with one boundary edge clamped or simply supported.

Boundary edge restrained	Clamped		Simply supported		Results by FEM [16]			
					Clamped		Simply supported	
Aspect ratio α	1.0	1.5	1.0	1.5	1.0	1.5	1.0	1.5
Natural frequency parameters λ								
First bending (1B)	3.4710	3.4534	0	0	3.4710	3.4535	0	0
First torsional (1T)	8.5066	11.6565	6.6437	9.8451	8.5088	11.6573	6.6457	9.8461
Second bending (2B)	21.2847	21.4660	14.9015	14.8874	21.3307	21.4889	14.9213	14.8989

Table 3
The minimum support stiffnesses and corresponding natural frequency parameters for the rectangular plate with a support at the mid-point of the free edge.

Boundary edge restrained	Clamped		Simply supported	Results by FEM [16]		
				Clamped		Simply supported
Aspect ratio, α	1.0	1.5	1.0	1.0	1.5	1.0
Natural frequency parameters λ						
1 (1B)	8.5066	11.6565	6.6437	8.5088	11.6573	6.6457
2 (1T)	8.5066	11.6565	6.6437	8.5088	11.6573	6.6457
3 (2B)	23.6962	27.5971	18.6904	23.7338	27.6186	16.1827
Minimum stiffness, γ_s	24.0239	47.9413	35.8290	23.9606	47.8070	35.7646

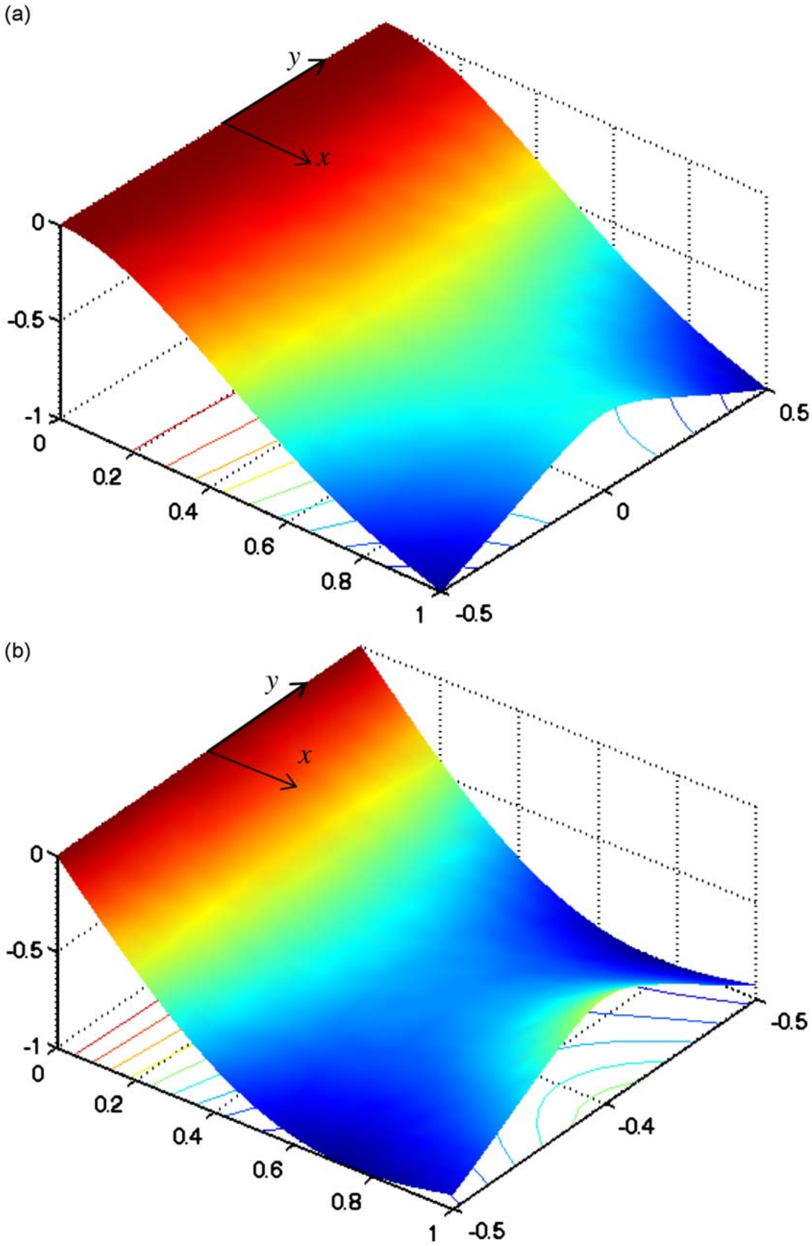

Fig. 2. The fundamental mode shape of the square plate supported by a single elastic support of the minimum stiffness on the free edge opposite to the restrained one: (a) with a clamped edge and (b) with a simply supported edge.

(lowest) natural frequency to the original first torsional (second) natural frequency at the present support location. Therefore, no solution is given in Table 3. This solution can also be proven with ease by considering the rectangular plate having two opposite edges simply supported. In this case the fundamental natural frequency parameter is equal to 9.5582 [18], less than the first torsional frequency parameter of the rectangular plate with an edge simply supported. Alternatively, to make the first frequency arrive at the original second natural frequency the single elastic support has to depart from the free edge of the plate (see Section 5.1.3).

5.1.2. Two elastic supports on the free edge opposite to the restrained edge

Let two elastic point supports with the same stiffness lie along the free edge to raise the natural frequency of the rectangular plate. To compare the results attained with one support, the fundamental natural frequency is firstly raised to the second natural frequency of the unsupported plate. Clearly, these two supports should be located at two symmetric points on the free edge about the *x*-axis, $(1, \eta_0)$ and $(1, -\eta_0)$ due to the symmetry of the first bending vibration mode. Fig. 3

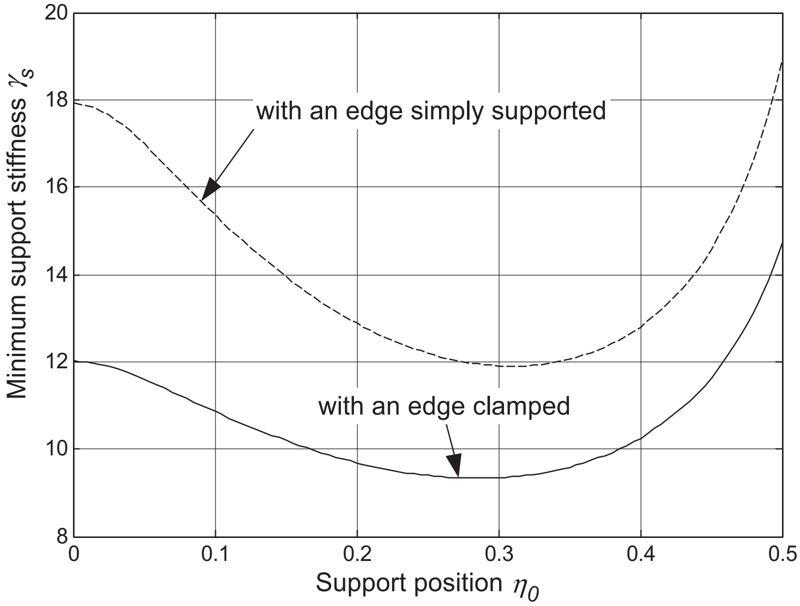

Fig. 3. Variations of the minimum support stiffness versus support position on the free edge for the square plate.

Table 4
The minimum support stiffnesses and optimal positions along with the corresponding natural frequency parameters for the rectangular plate with two symmetric supports on the free edge.

Boundary edge restrained	Clamped		Simply supported	Results by FEM [16]		
				Clamped		Simply supported
Aspect ratio, α	1.0	1.5	1.0	1.0	1.5	1.0
Natural frequency parameters λ						
1 (1B)	8.5066	11.6565	6.6437	8.5088	11.6573	6.6457
2 (1T)	11.0134	16.3653	9.7549	10.9957	16.4054	9.7728
Minimum stiffness, γ_s	9.3279	18.3182	11.8959	9.3262	18.2840	11.8846
Optimal position, η_b	± 0.2855	± 0.3136	± 0.3087	± 0.284	± 0.316	± 0.310

shows, respectively, the variation of the minimum support stiffness as a function of the support position η_0 for a square plate having a boundary edge restrained differently. Apparently, once the support departs from the midpoint of the free edge, the minimum stiffness begins to decrease steadily, but then increases sharply while approaching to the free corners. In Table 4 both the minimum support stiffnesses and the optimal positions η_b are presented along with the results estimated by using the FEM [16]. The total minimum support stiffnesses, in each case, are clearly less than those obtained in Section 5.1.1. In addition, it is noteworthy that the respective first natural frequency is not yet repeated in this situation because the two point supports are not located on the nodal line of the first torsional mode shape. Fig. 4 shows, respectively, the two fundamental vibration modes of a square plate restrained differently with the minimum stiffness supports at the optimal positions. However, for the same reason discussed before, there is no solution for the simply supported rectangular plate with aspect ratio 1.5.

With the two elastic supports, attempts are made to raise the fundamental natural frequency of the plate to the third (the second bending) natural frequency of the unsupported plate, given in Table 2. Unsurprisingly, there is no positive eigenvalue for this exploration. Therefore, we understand that with the supports on the free edge opposite to the restrained edge of the rectangular plate, it is impossible to attain the lowest natural frequency being equal to the third natural frequency of the unsupported plate.

5.1.3. *One elastic support on the center line*

It has already been addressed previously that one or two rigid supports at the free edge opposite to the restrained edge of the rectangular plate with aspect ratio α=1.5 are not able to raise the fundamental natural frequency to the second natural frequency of the unsupported plate. In order to accomplish this specific frequency increment, the supports have to move away from the free edge [14]. Suppose that a single flexible support is placed along the axis of symmetry with η_0=0

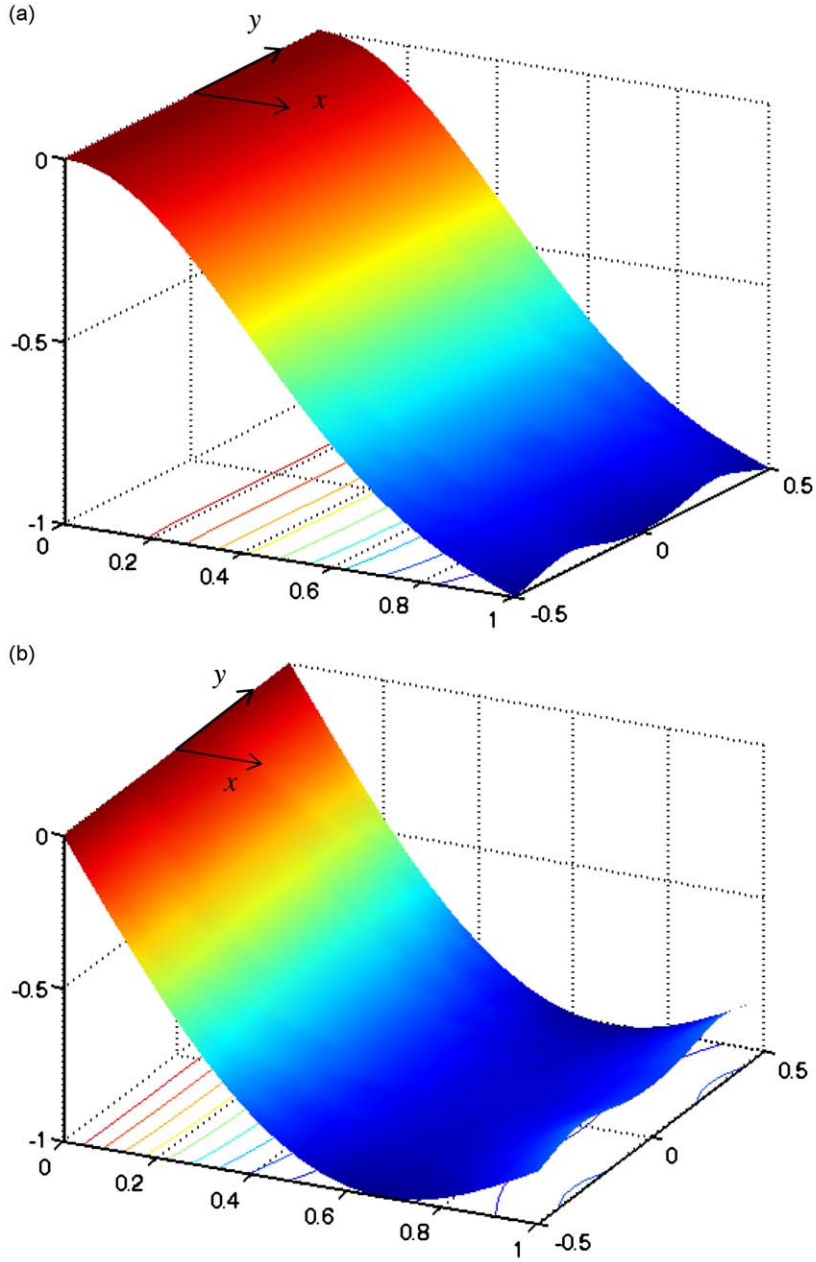

Fig. 4. The fundamental mode shape of the square plate supported by two elastic supports of the minimum stiffness on the free edge opposite to the restrained one: (a) with a clamped edge and (b) with a simply supported edge.

so that the requirement of the zero slope in the *y*-direction is met in advance. Fig. 5 shows the variation of the minimum stiffness versus the support position ξ_0 for the rectangular plate with a boundary edge restrained differently. It is impressive that for the targeted fundamental natural frequency, the support stiffness becomes increasingly large when a point support is close to the locations ξ_0=0.55 or 0.96 for the plate with an edge simply supported at ξ_0=0. Beyond this region, a rigid support is almost unable to raise the first frequency sufficiently. On the other hand, the minimum stiffness for the cantilever plate increases rapidly when the location is only close to the ξ_0=0.58. Fig. 5 also indicates that the minimum stiffness occurs approximately at a support location ξ_0=0.79 for the plate of a simply supported edge, and ξ_0=0.90 for the cantilever plate. Obviously, the optimal position moves farther away from the free edge with a less restrained edge of the plate. In Table 5 the first two natural frequency parameters, optimal support location and minimum stiffness are given for the rectangular plate. For completeness Table 5 also shows the results for a square plate of an edge restrained differently and those achieved by FEM [16]. Comparison shows good agreement between the two methods with the differences of the minimum stiffness less than 0.4%. Furthermore, for a cantilever square plate Narita [1] had illustrated

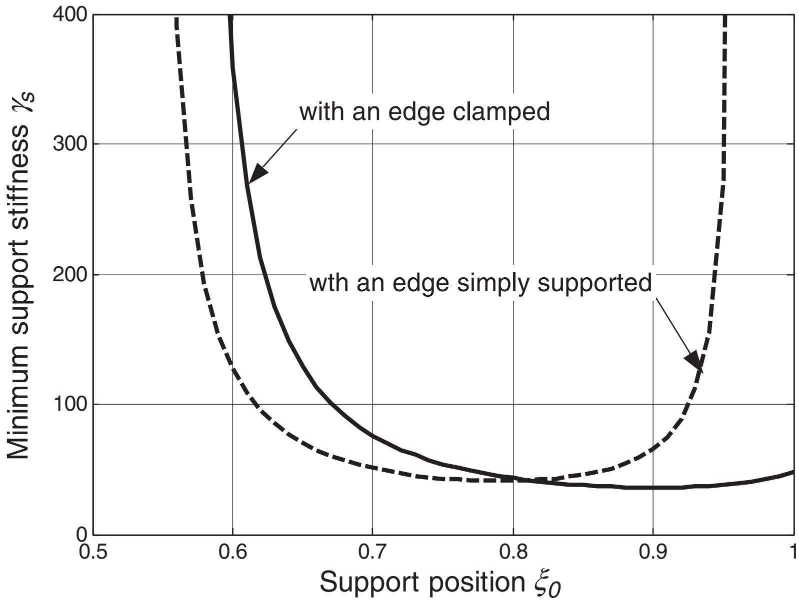

Fig. 5. Variations of the minimum support stiffness versus support position along the center line for the rectangular plate of aspect ratio 1.5.

Table 5
The minimum support stiffnesses and optimal positions along with the corresponding natural frequency parameters for the rectangular plate with a support on the plate center line.

Boundary edge restrained	Clamped		Simply supported		Results by FEM [16]			
					Clamped		Simply supported	
Aspect ratio, α	1.0	1.5	1.0	1.5	1.0	1.5	1.0	1.5
Natural frequency parameters, λ								
1 (1B)	8.5066	11.6565	6.6437	9.8451	8.5088	11.6573	6.6457	9.8461
2 (1T)	8.5066	11.6565	6.6437	9.8451	8.5088	11.6573	6.6457	9.8461
Minimum stiffness, γ_s	23.6588	35.9539	26.1363	41.1650	23.6313	36.0017	26.2139	41.2976
Optimal position, ξ_b	0.9720	0.9012	0.8710	0.7913	0.9734	0.9017	0.8711	0.7917

that a rigid point support can raise the fundamental natural frequency to the second one in the region ξ_0=0.5–1.0 along the plate centerline. But no details about the support properties were presented. Fig. 6 shows the two symmetric fundamental mode shapes of the rectangular plate restrained differently with the optimal support design, respectively, and Fig. 7 shows the two antisymmetric fundamental mode shapes of the rectangular plate due to the coalescence of the first two frequencies.

Close examination of Tables 2 and 5 illustrates that with increase in the aspect ratio α, the difference of the first two frequencies of the unsupported plate increases. As a result, the required minimum stiffness of the point support increases and the (nondimensional) optimal position moves farther away from the free boundary. Moreover, comparing corresponding results in Table 3 and in Table 5, it is interesting to find out that the minimum support stiffness decreases remarkably when the support location moves apart from the free edge, especially for the plate with an edge simply supported. For example, the minimum support stiffness decreases by 27.1% for the square plate. Therefore, we recognize that the location at the free edge is not optimum for the plate when raising the fundamental natural frequency. In view of the slope at the support location, this fact can also be highlighted by the non-zero slope of the mode in the x-direction on the free edge when we compare the corresponding vibration shapes in Fig. 2 with Fig. 6, where the slope at the support location does vanish completely [16].

5.2. Two adjacent edges of the rectangular plate restrained

5.2.1. One elastic support along the diagonal

The model under study is a square plate with two adjacent edges restrained similarly, and the others are completely free. Suppose that a single flexible support is placed along the symmetric diagonal to raise the fundamental natural frequency (corresponding to the first bending mode) to the second frequency (corresponding to the first torsional mode) of

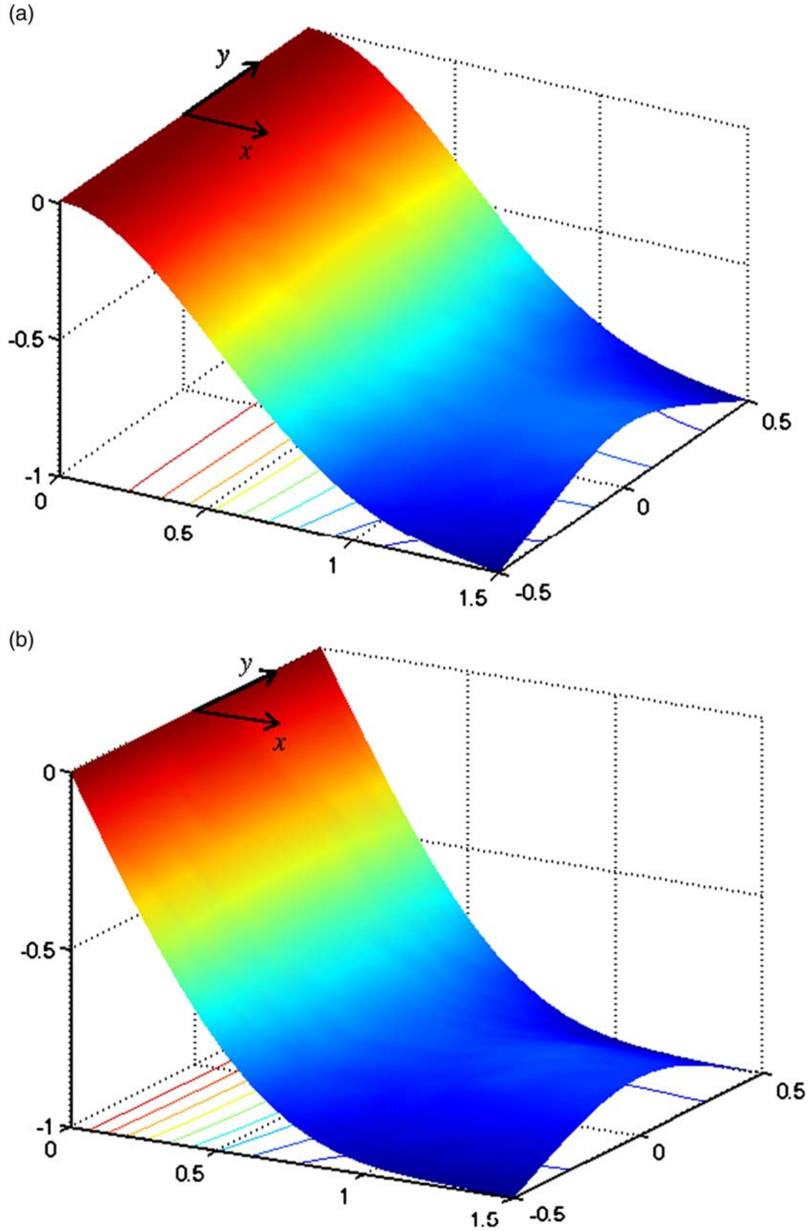

Fig. 6. The symmetric fundamental mode shape of the rectangular plate of aspect ratio 1.5 supported by an elastic support of the minimum stiffness on the plate center line: (a) with a clamped edge and (b) with a simply supported edge.

the unsupported plate. Obviously, the symmetric diagonal is the nodal line of the second vibration mode. The configuration of the plate is shown in Fig. 8 together with the coordinate system. The first three natural frequency parameters of the unsupported plate with the boundary edges clamped or simply supported are listed, respectively, in the first two columns in Table 6 by using the Rayleigh–Ritz method. In the last two columns are given the first three natural frequencies of the supported plate together with the minimum stiffnesses and optimal support positions. Obviously, the first two natural frequencies coalesce and become a double frequency with the optimal support design, and the fundamental natural frequency arrives at its upper limit. Fig. 9 illustrates, respectively, the slopes of the fundamental mode shape in the x-direction at the support point with variation of the support locations (due to symmetry, $\theta_x = -\theta_y$). At the optimal location, approximately, $\xi_0 = 0.73$ for the simply supported and 0.78 for the clamped edges, the zero-slope requirements along the two axes are satisfied simultaneously. Likewise, the optimal position moves farther away from the free edge in accordance with the descent of the degree of the restrained plate edges.

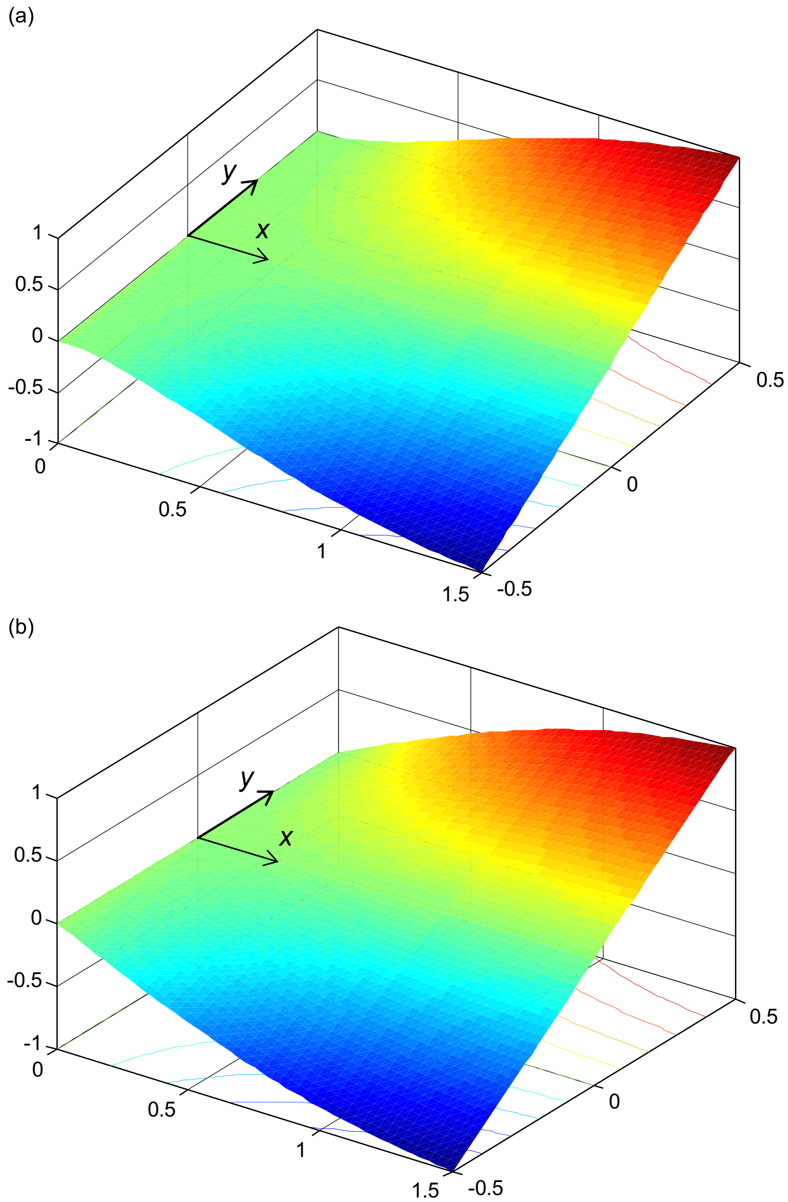

Fig. 7. The antisymmetric fundamental mode shape of the rectangular plate of aspect ratio 1.5 supported by an elastic support of the minimum stiffness on the plate center line: (a) with a clamped edge and (b) with a simply supported edge.

5.2.2. Two elastic supports along the free edges

Suppose two elastic supports, placed along the remaining free edges of the rectangular plate, are employed to increase the fundamental natural frequency to the original second frequency of the plate. In this case of the two adjacent edges restrained similarly, the second natural frequency is no longer corresponding to the first torsional mode for the rectangular plate of aspect ratio α=1.5. Given that the two supports have the same stiffness, the support locations will be symmetric about the diagonal for the square plate, but asymmetric for the plate of aspect ratio 1.5. Thus, the two optimal support positions (ξ_b, 1) and (1, η_b) need determining separately although they are highly correlated. Table 7 presents the computational results for the plate with the edges clamped or simply supported before and after supporting by the elastic supports. Now, the first two natural frequencies of the supported plate with the optimal support solution are not coalescent with each other, and the second frequency has also been raised. However, the increment is much smaller than that of the fundamental natural frequency, especially for the square plate.

It is noticed from Table 7 that for the rectangular plate of aspect ratio α=1.5, the minimum stiffness of the double supports is much smaller than that for the square plate. This is because the second natural frequency of the unsupported rectangular plate is virtually associated with the second bending mode, not with the first torsional mode for the square

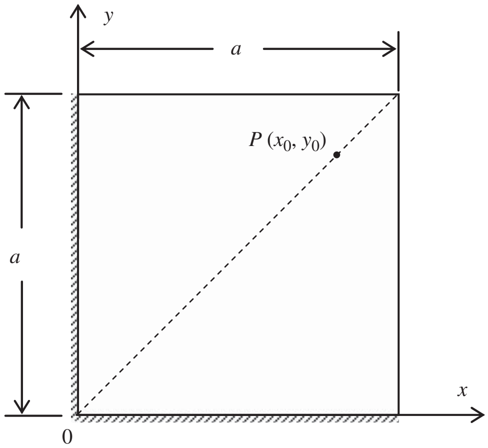

Fig. 8. A uniform square plate with two adjacent edges conventionally restrained is additionally supported by a single elastic point support on the symmetric diagonal.

Table 6
The minimum support stiffnesses and optimal positions along with the corresponding natural frequency parameters for a square plate of two adjacent boundary edges clamped or simply supported with a support on the plate diagonal.

Boundary edge restrained	Unsupported		Supported	
	Clamped	Simply supported	Clamped	Simply supported
Natural frequency parameters, λ				
1 (1B)	6. 9194	3.3670	23.9036	17.3164
2 (1T)	23.9036	17.3164	23.9036	17.3164
3 (2B)	26.5850	19.2929	27.3237	19.6414
Minimum stiffness, γ_s			191.9712	157.8885
Optimal position, $\xi_b(=\eta_b)$			0.7810	0.7335

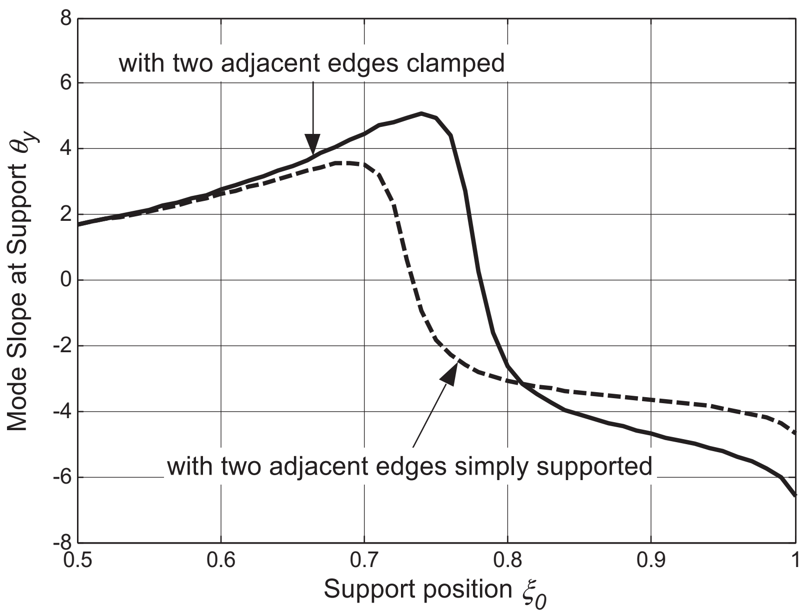

Fig. 9. Variations of the slope of the symmetric mode shape versus support position along the diagonal for the square plate with two adjacent edges restrained differently.

Table 7
The minimum support stiffnesses and optimal positions along with the corresponding natural frequency parameters for a rectangular plate of two adjacent boundary edges clamped or simply supported with two supports on the plate free edges.

Boundary edge restrained	Unsupported				Supported			
	Clamped		Simply supported		Clamped		Simply supported	
Aspect ratio, α	1.0	1.5	1.0	1.5	1.0	1.5	1.0	1.5
Natural frequency parameters, λ								
1	6. 9194	11.1778	3.3670	5.0239	23.9036	29.7781	17.3164	21.4637
2	23.9036	29.7781	17.3164	21.4637	28.8113	42.3085	20.4895	32.2737
3	26.5850	52.3718	19.2929	37.5273	41.3512	58.2571	34.3010	42.4797
Minimum stiffness, γ_s					204.6137	76.4098	206.4182	74.4455
Optimal positions, ξ_b					0.6260	0.6217	0.5444	0.5576
(ξ_b, 1) and (1, η_b), η_b					0.6260	0.8290	0.5444	0.7143

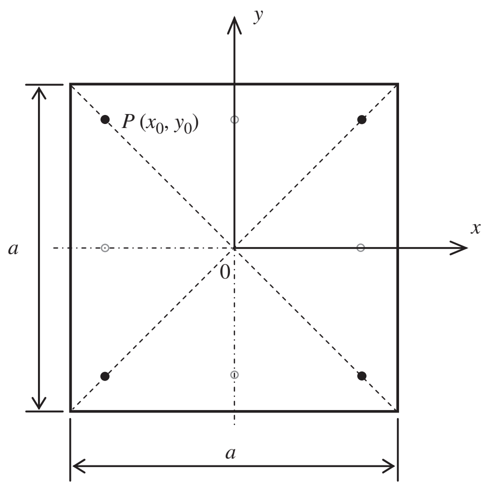

Fig. 10. A uniform square plate of fully free edges is supported by four elastic point supports on the diagonals (full points) or on the axes (hollow points).

plate. Moreover, it is seen by comparison between Tables 6 and 7 that for the square plate the minimum stiffnesses of the double supports are much larger than those of one point support on the diagonal. In view of raising the fundamental natural frequency to the second frequency, this fact indicates evidently that one support on the symmetric diagonal is much more efficient just because its location is on the nodal line of the related vibration mode.

5.3. *A free square plate*

The final analysis model is a fully free square plate without any restraint on the boundary edges, as shown in Fig. 10 along with the new coordinate system. Four identical elastic supports, located symmetrically along the plate diagonals, are utilized to increase the fundamental natural frequency. Table 8 lists the first three non-zero natural frequency parameters of the unsupported plate. In addition, the plate has six zero frequencies corresponding to rigid-body modes. Wang et al. [15] have demonstrated that with rigid point supports the fundamental natural frequency can be raised ultimately to its upper limit of the second flexural frequency of the unsupported plate. Therefore, in this study the objectives of increments of the fundamental natural frequency are, respectively, the original first and the second flexural frequencies of the plate. The final results of the optimal solutions of the supports are given in Table 8 along with the result estimated by FEM [15]. With the respective optimal support solution, the fundamental natural frequency becomes a doubly repeated frequency for the desired frequency parameter 13.4682, and a triply repeated frequency for the desired frequency parameter 19.5961. The optimal support location is near 0.29 for the two objective frequencies, which is in perfect agreement with the early estimation [15]. Fig. 11 shows two typical basis modes corresponding to each of the fundamental natural frequency of the plate with the minimum stiffness supports. For the fundamental frequency parameter λ_1=13.4682, the flexural deformation of the plate is comparatively very small. The main part of the basis mode is the deformation of the supports together with the rigid motion of the plate. On the other hand, except for the two modes of the main rigid-motion of the plate similar to that in Fig. 11a, another one of the basis modes is closely related to the flexural deformation of the plate for the fundamental frequency parameter λ_1=19.5961 (see Fig. 11b).

Table 8
The minimum support stiffnesses and optimal positions together with the corresponding natural frequency parameters for a fully free square plate with four supports on the diagonals or axes.

Support layout	Unsupported	Supports on the diagonals		Supports on the axes	Results by FEM [15] Supported on the diagonals
Objective frequency		First	Second	First	Second
Natural frequency parameters, λ					
1	13.4682	13.4682	19.5961	13.4682	19.5963
2	19.5961	13.4682	19.5961	13.4682	
3	24.2702	13.4722	19.5961	13.4682	
Minimum stiffness, γ_s		48.8639	116.9779	58.3136	118.1421
Optimal position, ($\xi_b = \pm\eta_b$)		0.2901	0.2892	0.4447	0.290

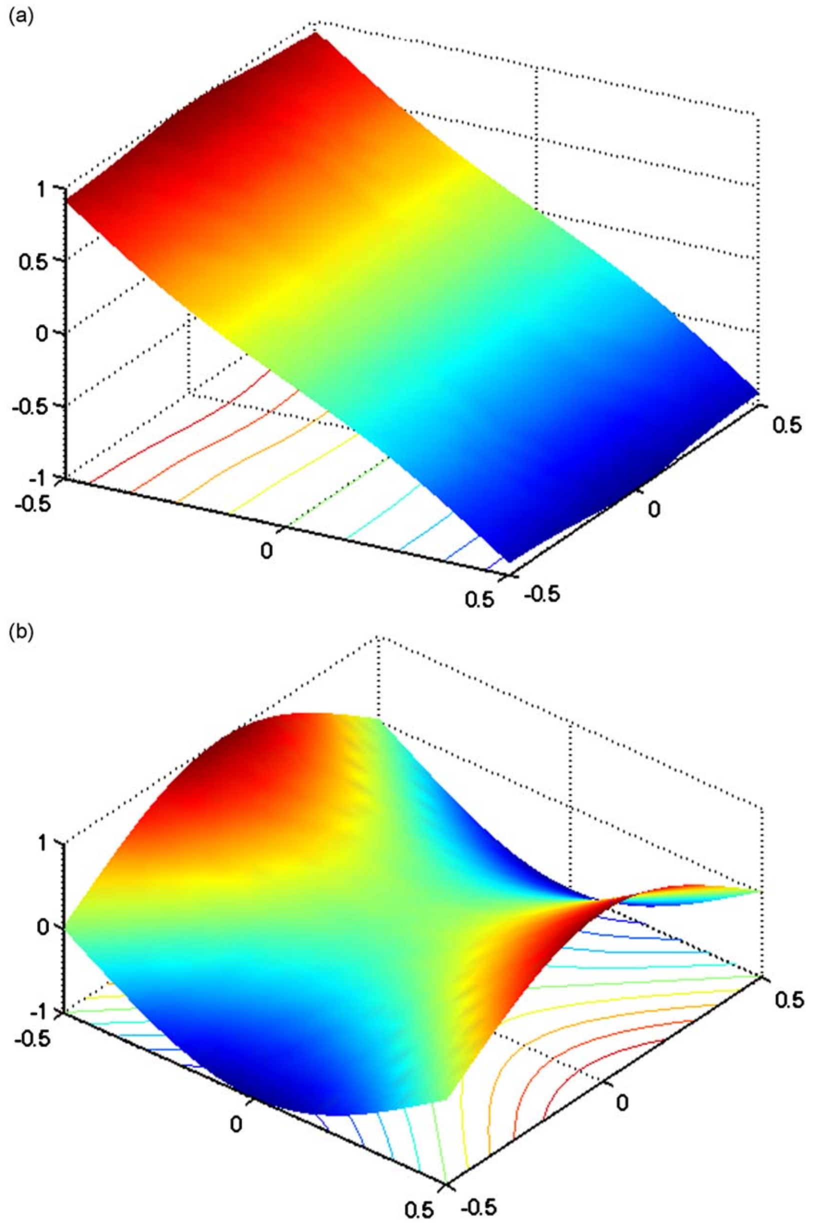

Fig. 11. One of the fundamental mode shapes of the square plate supported by four elastic supports of the minimum stiffness on the diagonals: (a) for the first flexural frequency and (b) for the second flexural frequency.

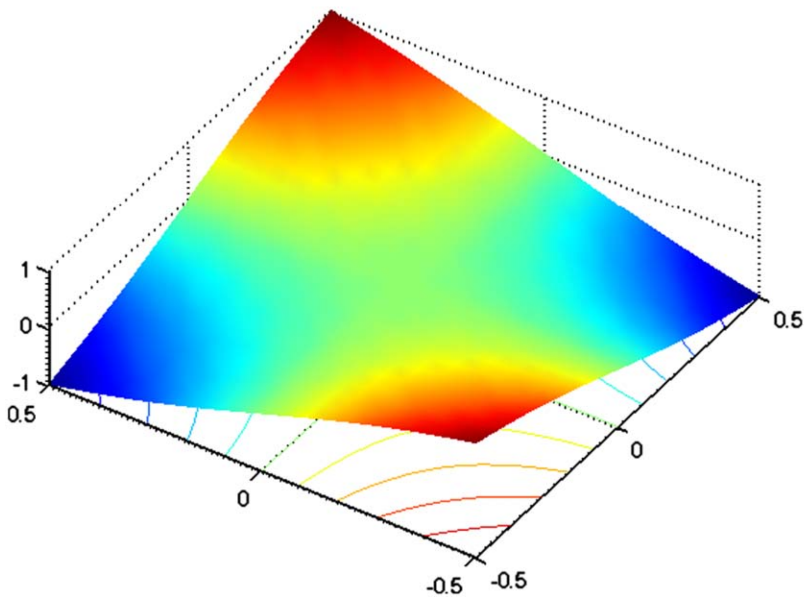

Fig. 12. The fundamental flexural mode shape of the square plate of the first flexural frequency supported by four elastic supports of the minimum stiffness on the axes.

Alternatively, for comparison of the optimal support solutions, we investigate further the plate with the four supports symmetrically located on the axes (the grey points in Fig. 10). The first three natural frequency parameters of the supported plate are also listed in Table 8 along with the optimal support location as well as the minimum stiffness. In this case, the fundamental natural frequency parameter of the plate can only be raised to 13.4682, which is also a triply repeated frequency. Moreover, the minimum stiffness is 19.3% larger than that with the supports located on the diagonals. Fig. 12 shows the basis flexural mode corresponding to the fundamental natural frequency of the plate with the optimal support solution. This solution can be simply verified since the supports are now just on the nodal lines of the corresponding mode shape. Even if the two axes are totally restrained with simple line supports, the fundamental natural frequency cannot be increased further. As the result, no solution is listed for the case of the objective frequency $\lambda_1 = 19.5961$. When comparing with Fig. 11a, it is known that for the same targeted fundamental natural frequency of the plate, the vibration mode shapes may be quite different with different layouts of the additional supports.

6. Conclusions

When a natural frequency of a plate needs to be increased, elastic point supports can almost achieve the same effect as rigid ones provided that they are located properly. In this work, the Rayleigh–Ritz method is applied to analyze the optimal configuration of the additional supports. Both the minimum stiffness and optimal position of the supports are obtained for raising the fundamental natural frequency of a rectangular plate to its upper limit of the second natural frequency of the unsupported plate. Numerical examples illustrate that the present procedure can work out the optimal support solutions directly and accurately. Since the Rayleigh–Ritz procedure is almost the same with the FEM except for the suitable regions of the shape functions, good agreements of the results can be anticipated between the two methods [15,16]. However, the size of the characteristic eigenvalue problem is much smaller in the present approach. So its efficiency is higher.

From the support optimization procedures of the typical examples in this paper, it is found that for increasing the fundamental natural frequency of a plate structure both the minimum stiffness and the optimal position of the support are highly dependent on the boundary conditions of the plate. Generally, for the similar rectangular plates of an edge restrained differently (simply supported or clamped) the optimal position tends to move toward the less restrained edge. Sometimes, there is no positive and finite solution in solving the characteristic eigenvalue equation. In this case even a rigid support cannot raise the fundamental natural frequency to the objective frequency. The numerical results also show that a plate with flexible supports may be designed with the same fundamental natural frequency as that with rigid point supports, even with the rigid line supports.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (10672135) and the Aeronautical Foundation, PR China (2007ZA53002).

References

[1] Y. Narita, Effect of point constraints on transverse vibration of cantilever plates, *Journal of Sound and Vibration* 102 (1985) 305–313.
[2] R.B. Bhat, Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh–Ritz method, *Journal of Sound and Vibration* 102 (1985) 493–499.
[3] C.S. Kim, P.G. Young, S.M. Dickinson, On the flexural vibration of rectangular plates approached by using simple polynomials in the Rayleigh–Ritz method, *Journal of Sound and Vibration* 143 (1990) 379–394.
[4] C.S. Kim, S.M. Dickinson, The flexural vibration of rectangular plates with position supports, *Journal of Sound and Vibration* 117 (1987) 249–261.
[5] R.B. Bhat, Vibration of rectangular plates on point and line supports using characteristic orthogonal polynomials in the Rayleigh–Ritz Method, *Journal of Sound and Vibration* 149 (1991) 170–172.
[6] L.T. Lee, D.C. Lee, Free vibration of rectangular plates on elastic point supports with the application of a new type of admissible function, *Computers & Structures* 65 (1997) 149–156.
[7] W.L. Li, Vibration analysis of rectangular plates with general elastic boundary supports, *Journal of Sound and Vibration* 273 (2004) 619–635.
[8] A.W. Leissa, The historical bases of the Rayleigh and Ritz methods, *Journal of Sound and Vibration* 287 (2005) 961–978.
[9] D.J. Gorman, A general-solution for the free-vibration of rectangular-plates resting on uniform elastic edge supports, *Journal of Sound and Vibration* 139 (1990) 325–335.
[10] H.T. Saliba, Free vibration analysis of rectangular cantilever plates with symmetrically distributed point supports along the edges, *Journal of Sound and Vibration* 94 (1984) 381–395.
[11] A.V. Bapat, N. Venkatramani, S. Suryanarayan, A new approach for representation of a point support in the analysis of plates, *Journal of Sound and Vibration* 120 (1988) 107–125.
[12] M.H. Huang, D.P. Thambiratnam, Free vibration analysis of rectangular plates on elastic intermediate supports, *Journal of Sound and Vibration* 240 (2001) 567–580.
[13] S.C. Fan, Y.K. Cheung, Flexural free vibrations of rectangular plates with complex support conditions, *Journal of Sound and Vibration* 93 (1984) 81–94.
[14] K.M. Won, Y.S. Park, Optimal support positions for a structure to maximize its fundamental natural frequency, *Journal of Sound and Vibration* 213 (1998) 801–812.
[15] D. Wang, J.S. Jiang, W.H. Zhang, Optimization of support positions to maximize the fundamental frequency of structures, *International Journal for Numerical Methods in Engineering* 61 (2004) 1584–1602.
[16] M.I. Friswell, D. Wang, The minimum support stiffness required to raise the fundamental natural frequency of plate structures, *Journal of Sound and Vibration* 301 (2007) 665–677.
[17] J.H. Son, B.M. Kwak, Optimization of boundary conditions for maximum fundamental frequency of vibrating structures, *AIAA Journal* 31 (1993) 2351–2357.
[18] A.W. Leissa, The free vibration of rectangular plates, *Journal of Sound and Vibration* 31 (1973) 257–293.